REALISATIONS OF QUANTUM $GL_{p,q}(2)$ AND JORDANIAN $GL_{h,h'}(2)$

DEEPAK PARASHAR and ROGER J. McDERMOTT

School of Computer and Mathematical Sciences, The Robert Gordon University, St. Andrew Street, Aberdeen AB25 1HG, United Kingdom
(e-mail: deeps@scms.rgu.ac.uk , rm@scms.rgu.ac.uk)

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The quantum group $GL_{p,q}(2)$ is known to be related to the Jordanian $GL_{h,h'}(2)$ via a contraction procedure. It can also be realised using the generators of the Hopf algebra $G_{r,s}$. We contract the $G_{r,s}$ quantum group to obtain its Jordanian analogue $G_{m,k}$, which provides a realisation of $GL_{h,h'}(2)$ in a manner similar to the q-deformed case.

1 Introduction

Non Standard (or Jordanian) deformations of Lie groups and Lie algebras has been a subject of considerable interest in the mathematical physics community. Jordanian deformations for $GL(2)$ were introduced in [1,2], its two parametric generalisation given in [3] and extended to the supersymmetric case in [4]. Non Standard deformations of $sl(2)$ (i.e. at the algebra level) were first proposed in [5], the universal R-matrix presented in [6-8] and irreducible representations studied in [9,10]. A peculiar feature of this deformation (also known as h-deformation) is that the corresponding R-matrix is triangular. It was shown in [11] that up to isomorphism, $GL_q(2)$ and $GL_h(2)$ are the only possible distinct deformations (with central determinant) of the group $GL(2)$. In [12], an interesting observation was made that the h-deformation could be obtained by a singular limit of a similarity transformation from the q-deformations of the group $GL(2)$. Given this contraction procedure, it would be useful to look for Jordanian deformations of other q-groups.

In the present paper, we focus our attention on a particular two parameter quantum group, denoted $G_{r,s}$, which provides a realisation of the well known $GL_{p,q}(2)$. We investigate the contraction procedure on $G_{r,s}$, in order to obtain its non standard counterpart. The generators of the contracted structure are employed to realise the two parameter non standard $GL_{h,h'}(2)$. This is similar to what happens in the q-deformed case.

2 Quantum $G_{r,s}$ and Realisation of $GL_{p,q}(2)$

The two parameter quantum group $G_{r,s}$ is generated by elements a, b, c, d, and f satisfying the relations

$$\begin{array}{ll} ab = r^{-1}ba, & db = rbd \\ ac = r^{-1}ca, & dc = rcd \\ bc = cb, & [a,d] = (r^{-1} - r)bc \end{array}$$

and

$$\begin{array}{ll} af = fa, & cf = sfc \\ bf = s^{-1}fb, & df = fd \end{array}$$

Elements a, b, c, d satisfying the first set of commutation relations form a subalgebra which coincides exactly with $GL_q(2)$ when $q = r^{-1}$. The matrix of generators is

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

and the Hopf structure is given as

$$\begin{array}{rcl} \Delta(T) & = & T\dot{\otimes}T \\ \varepsilon(T) & = & \mathbf{1} \end{array}$$

The Casimir operator is defined as $\mathbf{D} = ad - r^{-1}bc$. The inverse is assumed to exist and satisfies $\Delta(\mathbf{D}^{-1}) = \mathbf{D}^{-1} \otimes \mathbf{D}^{-1}$, $\varepsilon(\mathbf{D}^{-1}) = 1$, $S(\mathbf{D}^{-1}) = \mathbf{D}$, which enables determination of the antipode matrix $S(T)$, as

$$S \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \mathbf{D}^{-1} \begin{pmatrix} d & -rc & 0 \\ -r^{-1}c & a & 0 \\ 0 & 0 & \mathbf{D}f^{-1} \end{pmatrix}$$

The quantum determinant $\delta = \mathbf{D}f$ is group-like but not central.

The quantum group $G_{r,s}$ was proposed in [13] as a particular quotient of the multiparameter q-deformation of $GL(3)$. The structure of $G_{r,s}$ is interesting because it contains the one parameter q-deformation of $GL(2)$ as a Hopf subalgebra and also gives a simple realisation of the quantum group $GL_{p,q}(2)$ in terms of the generators of $G_{r,s}$. There is a Hopf algebra morphism $\mathcal{F}$ from $G_{r,s}$ to $GL_{p,q}(2)$ given by

$$\mathcal{F} : G_{r,s} \longmapsto GL_{p,q}(2)$$

$$\mathcal{F} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = f^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The elements a',b',c' and d' are the generators of $GL_{p,q}(2)$ and N is a fixed non-zero integer. The relation between the deformation parameters (p, q) and (r, s) is given by

$$p = r^{-1}s^N \quad , \qquad q = r^{-1}s^{-N}$$

This quantum group can, therefore, be used to realise both $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups.

3 *R*-matrices and Contraction limits

The R- matrix of $G_{r,s}$ explicitly reads

$$R(G_{r,s}) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r-r^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & r-r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & r-r^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

with entries labelled in the usual numerical order (11), (12), (13), (21), (22), (23), (31), (32), (33). If we reorder the indices of this R-matrix with the elements in the order (11), (12), (21), (22), (13), (23), (31), (32), (33), then we obtain a block matrix, say R_q which is similar to the form of the $GL_q(2)$ R-matrix with the q in the R^{11}_{11} position itself replaced by the $GL_q(2)$ R-matrix.

$$R_q = \begin{pmatrix} R(GL_r(2)) & 0 & 0 & 0 \\ 0 & S & \lambda I & 0 \\ 0 & 0 & S^{-1} & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

where $R(GL_r(2))$ is the 4×4 R-matrix for $GL_q(2)$ with $q = r$, $\lambda = r - r^{-1}$, I is the 2×2 identity matrix and S is the 2×2 matrix $S = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ where r and s are the deformation parameters. The zeroes are the zero matrices of appropriate order. The usual block structure of the R-matrix is clearly visible in this form.

It is well known [12] that the non standard R-matrix $R_h(2)$ can be obtained from the q-deformed $R_q(2)$ as a singular limit of a similarity transformation

$$R_h(2) = \lim_{q\to 1}(g^{-1}\otimes g^{-1})R_q(2)(g\otimes g)$$

where $g = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$. Such a transformation has been generalised to higher dimensions [14] and has also been successfully applied to two parameter quantum groups. Here we apply the above transformation for the $G_{r,s}$ quantum group. Our starting

point is the block diagonal form of the $G_{r,s}$ R-matrix, denoted R_q

$$R_q = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

where $\lambda = r - r^{-1}$. We apply to R_q the transformation

$$(G^{-1} \otimes G^{-1}) R_q (G \otimes G)$$

Here the transformation matrix G is a 3×3 matrix and chosen in the block diagonal form

$$G = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

where g is the transformation matrix for the two dimensional case. Substituting $\eta = \frac{m}{r-1}$ and then taking the singular limit $r \to 1$, $s \to 1$ (such that $\frac{1-s}{1-r} \to \frac{k}{m}$) yields the Jordanian R-matrix

$$R_h = R(G_{m,k}) = \begin{pmatrix} 1 & m & -m & m^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the entries are labelled in the block diagonal form (11), (12), (21), (22), (13), (23), (31), (32), (33). It is straightforward to verify that this R-matrix is triangular and a solution of the Quantum Yang Baxter Equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

It is interesting to note that the block diagonal form of $R(G_{m,k})$ embeds the R-matrix for the single parameter deformed $GL_h(2)$ for $m = h$.

4 Jordanian $G_{m,k}$ and Realisation of $GL_{h,h'}(2)$

A two parameter Jordanian quantum group, denoted $G_{m,k}$, can be formed by using the contracted R-matrix $R(G_{m,k})$ in conjunction with a T-matrix of the form

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

The RTT- relations give the commutation relations between the generators a, b, c, d and f.

$$\begin{array}{ll} [c,d] = mc^2, & [c,b] = m(ac+cd) = m(ca+dc) \\ [c,a] = mc^2, & [d,a] = m(d-a)c = mc(d-a) \end{array}$$

$$[d,b] = m(d^2 - D)$$
$$[b,a] = m(D - a^2)$$

and

$$\begin{array}{ll} [f,a] = kcf, & [f,b] = k(df - fa) \\ [f,c] = 0, & [f,d] = -kcf \end{array}$$

The element $D = ad - bc - mac = ad - cb + mcd$ is central in the whole algebra. The coalgebra structure of $G_{m,k}$ can be written as

$$\begin{array}{rcl} \Delta(T) & = & T\dot{\otimes}T \\ \varepsilon(T) & = & \mathbf{1} \end{array}$$

Adjoining the element D^{-1} to the algebra enables determination of the antipode matrix $S(T)$,

$$S\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = D^{-1}\begin{pmatrix} d - mc & -b - m(d-a) + m^2c & 0 \\ -c & a + mc & 0 \\ 0 & 0 & Df^{-1} \end{pmatrix}$$

(The Hopf structure of D^{-1} is $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \varepsilon(D^{-1}) = \mathbf{1}, S(D^{-1}) = D$.)

It is evident that the elements a, b, c and d of $G_{m,k}$ form a Hopf subalgebra which coincides with non standard $GL(2)$ with deformation parameter m. This is exactly analogous to the q-deformed case where the first four elements of $G_{r,s}$ form the $GL_q(2)$ Hopf subalgebra. Again, the remaining fifth element f generates the $GL(1)$ group, as it did in the q-deformed case, and the second parameter appears only through the cross commutation relations between $GL_m(2)$ and $GL(1)$ elements. Therefore, $G_{m,k}$ can also be considered as a two parameter Jordanian deformation of classical $GL(2) \otimes GL(1)$ group.

Now we wish to explore the connection of $G_{m,k}$ with the two parameter Jordanian $GL_{h,h'}(2)$. A Hopf algebra morphism

$$\mathcal{F} : G_{m,k} \longmapsto GL_{h,h'}(2)$$

of exactly the same form as in the q-deformed case, exists between the generators of $G_{m,k}$ and $GL_{h,h'}(2)$ provided that the two sets of deformation parameters (h, h') and (m, k) are related via the equation

$$h = m + Nk \quad , \qquad h' = m - Nk$$

Note that for vanishing k, one gets the one parameter case. In addition, using the above realisation together with the coproduct, counit and antipode axioms for the $G_{m,k}$ algebra and the respective homeomorphism properties, one can easily recover the standard coproduct, counit and antipode for $GL_{h,h'}(2)$. Thus, the non standard $GL_{h,h'}(2)$ group can in fact be reproduced from the newly defined non standard $G_{m,k}$. It is curious to note that if we write $p = e^h$, $q = e^{h'}$, $r = e^{-m}$ and $s = e^k$, then the relations between the parameters in the q-deformed case and the h-deformed case are identical.

5 Conclusions

We have applied the contraction procedure to the $G_{r,s}$ quantum group and obtained a new Jordanian quantum group $G_{m,k}$. The group $G_{m,k}$ has five generators and two deformation parameters and contains the single parameter $GL_h(2)$ as a Hopf subalgebra. Furthermore, we have given a realisation of the two parameter $GL_{h,h'}(2)$ through the generators of $G_{m,k}$ which also reproduces its full Hopf algebra structure. The results match with the q-deformed case.

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